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Relativistic theory of nucleons in laser fields

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Abstract. The Dirac equation for a particle with spin $\frac{1}{2}$ and an anomalous magnetic moment in an external wave field is reduced to a set of coupled ordinary differential equations for a spin factor, in which the wavefunction differs from the corresponding one for a 'normal' particle. Exact solutions are given for a linearly polarized laser wave of finite length and for an infinitely extended plane wave with circular polarization.

1. Introduction

The behaviour of particles in laser fields is of some interest, since these fields are strong. So far the investigations have been focused on electrons. If the laser field is treated as an external wave field, the corresponding Dirac equation for the wavefunction can be solved exactly (Volkov 1935) as long as the wave fronts of the field are planes. The wavefunction is then used in further investigations on physical effects, like eg Compton scattering, pair creation etc. Nucleons are influenced by electromagnetic fields via the charge and the (anomalous) magnetic moment. In spite of the fact that the magnetic interactions produce very small effects due to the magnitude of the nuclear magneton, one might ask whether very strong fields could lead to visible consequences. As a basis the corresponding wavefunction for the particle in the presence of the field is needed. We shall calculate this wavefunction without treating the interaction with the laser as a small perturbation. We shall, however, assume that the particle can, in this context, be characterized by its static properties (charge and magnetic moment) alone. This seems to be a reasonable assumption, as long as the momentum transfer between the particle and the field is small. Some plausibility arguments for this assumption are discussed in the appendix.

2. Reduction of the Dirac equation

The laser field will be described by a plane wave with wavevector

$$k^{\mu} = \omega(1, \boldsymbol{n}) \qquad \omega = 2\pi/\lambda$$

characterized by the vector potential

$$A^{\mu}(x) = ae_i{}^{\mu}A_i(\xi). \tag{1}$$

Here a denotes an intensity parameter and $e_i{}^{\mu}$ $(i = 1, 2)$ are polarization vectors. We have

$$k_{\mu}e_i{}^{\mu} = 0, \qquad e_i{}^{\mu}e_{j\mu} = -\delta_{ij} \tag{2}$$

and a summation convention $\Sigma_{i=1}^{2}$ is adopted for repeated latin indices. A_i are arbitrary functions of their argument $\xi = k^\mu x_\mu$ so that eg laser pulses of finite 'duration' are admitted, for which A_i vanishes for ξ outside some finite domain. The field tensor is

$$F^{\mu\nu}(x) = \partial^\mu A^\nu - \partial^\nu A^\mu = aF_i(\xi)f_i^{\mu\nu} \tag{3}$$

with

$$f_i^{\mu\nu} = k^\mu e_i^{\ \nu} - k^\nu e_i^{\ \mu}, \qquad F_i(\xi) = \frac{\mathrm{d}A_i(\xi)}{\mathrm{d}\xi}. \tag{4}$$

The Dirac equation for a nucleon reads (cf Bjorken and Drell 1964; we shall use the γ matrices, and the metric of this reference)

$$[\gamma_\mu(\mathrm{i}\partial^\mu - \epsilon a e_i^{\ \mu} A_i(\xi)) + gF_i(\xi)\tfrac{1}{2}\sigma_{\mu\nu}f_i^{\mu\nu} - \kappa]\psi = 0. \tag{5}$$

Here we have introduced

$$\epsilon = \frac{e}{\hbar c}, \qquad g = -\frac{ea}{2mc^2}\mu, \qquad \kappa = \frac{mc}{\hbar} \tag{6}$$

where $e = |e|$ is the elementary charge, m the nucleon mass and μ the anomalous magnetic moment of the particle. For the proton we have $\mu = 1{\cdot}79$, for the neutron we have to put $\mu = -1{\cdot}91$ and $\epsilon = 0$.

We introduce light-like components (Rohrlich 1971) using the *vierbein*

$$n^\mu = \frac{1}{\sqrt{(2)}\omega}k^\mu = \frac{1}{\sqrt{2}}(1, \boldsymbol{n}), \qquad \hat{n}^\mu = \frac{1}{\sqrt{2}}(1, -\boldsymbol{n}), \qquad e_i^\mu \tag{7}$$

and represent an arbitrary vector p^μ by

$$p^\mu = n^\mu p_u + \hat{n}^\mu p_v + e_i^\mu p_i \tag{8}$$

in terms of the components

$$p_u = \hat{n}^\mu p_\mu, \qquad p_v = n^\mu p_\mu, \qquad p_i = -e_i^{\ \mu} p_\mu.$$

With the abbreviations

$$x_v = u, \qquad x_u = v, \qquad A_i(\xi) = a_i(u), \qquad F_i(\xi) = \frac{1}{\sqrt{(2)}\omega}f_i(u) \tag{9}$$

the Dirac equation takes the form

$$\left[\mathrm{i}\gamma_v\frac{\partial}{\partial u} + \mathrm{i}\gamma_u\frac{\partial}{\partial v} + \gamma_i\left(\mathrm{i}\frac{\partial}{\partial x_i} + \epsilon a a_i(u)\right) + \mathrm{i}g f_i(u)\gamma_i\gamma_v - \kappa\right]\psi = 0. \tag{10}$$

If we imagine the laser to be switched on at $u = u_0$, the solution has to reduce to the free plane-wave solution for $u \leqslant u_0$. Therefore we use the ansatz

$$\psi(x|p) = M(u|p)\psi_0 \exp[-\mathrm{i}(u_0 p_u^{\ f} + vp_v - x_i p_i)]. \tag{11}$$

Here ψ_0 is a constant spinor and the exponential is the usual plane-wave expression with the constant vector $p^\mu = (p_u^{\ f}, p_v, p_i)$ on the mass shell

$$p_u^{\ f} = \frac{1}{2p_v}(\kappa^2 + p_i p_i). \tag{12}$$

The Dirac matrix M remains to be determined. If we introduce

$$M_u = \gamma_u M, \qquad M_v = \gamma_v M, \qquad \pi_i(p, u) = p_i - \epsilon a a_i(u) \tag{13}$$

we obtain the equation

$$p_v M_u + \mathrm{i}\left(\frac{\partial}{\partial u} + g\gamma_i f_i(u)\right) M_v - (\gamma_i \pi_i + \kappa) M = 0. \tag{14}$$

The reduction to scalar equations is most easily carried out by means of a projection technique (Rohrlich 1971) using the operators

$$\mathbb{P}_u = \tfrac{1}{2}\gamma_v\gamma_u, \qquad \mathbb{P}_v = \tfrac{1}{2}\gamma_u\gamma_v. \tag{15}$$

If we observe

$$\mathbb{P}_u M_v = M_v, \qquad \mathbb{P}_v M_u = M_u, \qquad \mathbb{P}_u M_u = \mathbb{P}_v M_v = 0$$

we obtain by application of $\mathbb{P}_v$ onto the Dirac equation (14)

$$M_u = \frac{1}{2p_v}(\gamma_i \pi_i + \kappa)\gamma_u M_v. \tag{16}$$

The entire matrix M can be expressed by M_v

$$M = (\mathbb{P}_u + \mathbb{P}_v) M = \frac{1}{2}(\gamma_v M_u + \gamma_u M_v) = \frac{1}{2p_v}(\gamma_u p_v - \gamma_i \pi_i + \kappa) M_v. \tag{17}$$

An equation for M_v is obtained by application of $\mathbb{P}_u$ onto equation (14) and insertion of relation (16). The result is

$$\left[\mathrm{i}\left(\frac{\partial}{\partial u} + g\gamma_i f_i\right) - \frac{1}{2p_v}(\kappa^2 + \pi_i\pi_i)\right] M_v = 0. \tag{18}$$

This matrix equation can be reduced if we observe that a complete basis in the space of the γ_i consists of the four elements

$$1, \qquad \gamma_i, \qquad \sigma_3 = \mathrm{i}\gamma_1\gamma_2 = \tfrac{1}{2}\mathrm{i}\epsilon_{kl}\gamma_k\gamma_l,$$

$$\epsilon_{12} = -\epsilon_{21} = 1, \qquad \epsilon_{11} = \epsilon_{22} = 0$$

and that a simplification is obtained by splitting off an appropriate exponential. Therefore we write

$$M_v = \gamma_v B(u) \exp[-\mathrm{i}(u - u_0) p_u(u, u_0)] \tag{19}$$

with

$$\begin{aligned} p_u(u, u_0) &= \frac{1}{u - u_0}\frac{1}{2p_v}\int_{u_0}^{u} (\kappa^2 + \pi_i(p, u')\pi_i(p, u'))\,\mathrm{d}u' \\ B(u) &= c_1(u) + \mathrm{i}\sigma_3 c_2(u) + \gamma_i b_i(u). \end{aligned} \tag{20}$$

Thus our entire solution has the form

$$\psi(x|p) = \frac{1}{2p_v}(\gamma_u p_v - \gamma_i \pi_i + \kappa)\gamma_v B(u)\psi_0 \exp\{-\mathrm{i}[(u - u_0)p_u(u, u_0) + v p_v - x_i p_i + u_0 p_u{}^f]\}. \tag{21}$$

The only difference from the Volkov solution for a particle without anomalous moment is the presence of the factor B involving the function c_i and b_i, which remains to be determined. If we insert our ansatz (19) into equation (18) and equate the coefficients of the basis elements, we obtain a linear system of differential equations, which does not contain p, so that B is independent of p. That this must be the case, is clear for physical reasons: the interaction of the field with the anomalous moment can only 'shake' the spin. It should also be noted that the exponential factor reduces to an ordinary plane wave for the neutron, since $p_u \to p_u^f$ in this case.

The system of equations for c_i, b_i reads

$$\begin{aligned} c_1' &= -g f_i b_i \\ c_2' &= -g f_i \epsilon_{ij} b_j \\ b_i' &= g(c_1 f_i + c_2 f_j \epsilon_{ji}). \end{aligned} \tag{22}$$

It can easily be seen that the quantity

$$K = c_i c_i + b_i b_i \tag{23}$$

is constant and that

$$\epsilon_{ij}(c_i' c_j - b_i' b_j) = 0. \tag{24}$$

The coupling between the four equations (22) can be further reduced. If we introduce

$$c_\pm = \frac{1}{\sqrt{2}}(c_1 \pm \mathrm{i} c_2) \tag{25}$$

(correspondingly for b and f) we obtain two separate systems

$$c_\pm' = -\sqrt{(2)} g f_\mp b_\pm, \qquad b_\pm' = \sqrt{(2)} g f_\pm c_\pm \tag{26}$$

from which we can deduce a second-order equation for $c_\pm$

$$c_\pm'' - c_\pm' \frac{f_\mp'}{f_\mp} + 2g^2 f_+ f_- c_\pm = 0. \tag{27}$$

The constant (23) reads in terms of the new functions (25)

$$K = 2(c_+ c_- + b_+ b_-) = 2 c_+ c_- \left[1 + \frac{1}{2g^2}\left(\frac{c_+'}{f_- c_+}\right)\left(\frac{c_-'}{f_+ c_-}\right)\right]. \tag{28}$$

Particular solutions of the system will be discussed later. We shall first investigate the normalization of the wavefunction, which turns out to be independent of c and b, and related questions.

3. Current vector and normalization

In order to discuss these problems simultaneously we shall consider the matrix element

$$\bar{\psi}(x|q)\Gamma\psi(x|p) = \mathrm{e}^{-\mathrm{i}\phi}\bar{\psi}_0 R(q, p|\Gamma)\psi_0 \tag{29}$$

where Γ is an arbitrary Dirac matrix and we have

$$\phi = (u-u_0)[p_u(u,u_0)-q_u(u,u_0)]+v(p_v-q_v)-x_i(p_i-q_i)+u_0(p_u^f-q_u^f)$$
$$R(q,p|\Gamma) = \frac{1}{4p_v q_v}\gamma_0[(\gamma_u q_v-\gamma_i\pi_i(q,u)+\kappa)\gamma_v B(u)]^\dagger\gamma_0\ \Gamma(\gamma_u p_v-\gamma_i\pi_i(p,u)+\kappa)\gamma_v B(u). \tag{30}$$

The Dirac matrices used here have the property

$$\gamma_0\gamma_\mu^\dagger\gamma_0 = \gamma_\mu, \qquad \gamma_0(\mathrm{i}\sigma_3)^\dagger\gamma_0 = -\mathrm{i}\sigma_3.$$

Since c_i, b_i are real, we can rewrite R in the form

$$R(q,p|\Gamma) = (c_1-\mathrm{i}\sigma_3c_2+\gamma_ib_i)\left(\mathbb{P}_u+\frac{1}{2q_v}(\gamma_i\pi_i(q,u)+\kappa)\gamma_v\right)$$
$$\times\Gamma\left(\mathbb{P}_v+\gamma_v\frac{1}{2p_v}(\gamma_i\pi_i(p,u)+\kappa)\right)(c_1+\mathrm{i}\sigma_3c_2+\gamma_ib_i). \tag{31}$$

This formula is useful for the computation of matrix elements in general.

For the components of the particle current

$$j^\mu(x) = \bar\psi(x|p)\gamma^\mu\psi(x|p)$$

we obtain, evaluating the corresponding matrix elements

$$(j_u, j_v, j_i) = \frac{K}{p_v}(\bar\psi_0\gamma_v\psi_0)\left(\frac{\kappa^2+\pi_i\pi_i}{2p_v}, p_v, \pi_i\right). \tag{32}$$

For the neutron this yields the free current. The normalization of the wavefunction has to be achieved in such a way that the appropriate boundary conditions for a laser pulse of finite extent (in u) are respected. We observe that the four vector

$$\mathscr{J}^\mu(x|p,q) = \bar\psi(x|q)\gamma^\mu\psi(x|p)$$

satisfies a continuity equation. The surface integral

$$N_{qp} = \int \mathscr{J}^\mu(x|p,q)\,\mathrm{d}\sigma_\mu(x)$$

is therefore independent of the hypersurface of integration. Thus we can take the surface u = constant, $\mathrm{d}\sigma_u = \mathrm{d}^2x_i\,\mathrm{d}v$ and obtain

$$N_{qp} = \int \mathscr{J}_v(x|q,p)\,\mathrm{d}^2x_i\,\mathrm{d}v = (2\pi)^3K\delta(p_v-q_v)\delta_2(p_i-q_i)(\bar\psi_0\gamma_v\psi_0). \tag{33}$$

Thus we should take $K = (2\pi)^{-3}$ for normalized ψ_0 in order to obtain the usual δ-function normalization for plane waves. The construction of wave packets can be done as usual (Neville and Rohrlich 1971).

4. Special solutions

In some special cases the differential equations for c and b can be solved analytically.

We consider at first a linearly polarized laser wave

$$f_1(u) = f(u), \qquad f_2 = 0. \tag{34}$$

In this case the system (22) can be transformed into a linear system with constant coefficients by introduction of a new variable

$$\eta = g\int_{u_0}^{u} f(u')\,\mathrm{d}u' = g\int_{\xi_0}^{\xi_1} F(\xi')\,\mathrm{d}\xi' = g(A_1(\xi)-A_1(\xi_0)) \tag{35}$$

in place of u. Here u_0 is the initial point, at which the laser is switched on. The general solution of equation (22) is

$$\begin{aligned} c_i &= \alpha_i \cos\eta + \beta_i \sin\eta \\ b_i &= \alpha_i \sin\eta - \beta_i \cos\eta \end{aligned} \tag{36}$$

where α_i, β_i are arbitrary integration constants to be fixed by the initial conditions. The constant K is

$$K = \alpha_i\alpha_i + \beta_i\beta_i. \tag{37}$$

We note that the case of a constant crossed field is contained in our formulae (34)–(36), if we take

$$A_i(\xi) = \begin{cases} \xi\delta_{i1} & \xi_0 \leqslant \xi \leqslant \xi_1 \\ 0 & \text{otherwise.} \end{cases}$$

In this case $a\omega$ has to be identified with the magnitude of the field strength and $\boldsymbol{n}$ is the direction of the Poynting vector. For $\xi \leqslant \xi_0$ the solution goes over into the free one. If a solution is wanted in which the field is always present we have to take $A_i = \xi\delta_{i1}$ for all values of ξ and omit the factor $\exp(-\mathrm{i}u_0 p_u^f)$. The constants (including the lower limit of the integration in equation (35)) have to be fixed by imposing initial conditions on some null plane $\xi = \xi_0$ inside the field, since the particle is never separated from the latter.

Another simple solution can be found for a plane wave laser with circular polarization

$$F_1(\xi) = -\sin\xi, \qquad F_2(\xi) = -\cos\xi. \tag{38}$$

The differential equations (26) have constant coefficients in this case and the general solution is

$$\begin{aligned} c_1 &= gB_1\cos\rho_1\xi + gC_1\sin\rho_1\xi + B_2\cos\rho_2\xi + C_2\sin\rho_2\xi \\ c_2 &= gB_1\sin\rho_1\xi - gC_1\cos\rho_1\xi + B_2\sin\rho_2\xi - C_2\cos\rho_2\xi \\ b_1 &= \rho_1(-B_1\cos\rho_2\xi + C_1\sin\rho_2\xi) + \frac{g}{\rho_1}(B_2\cos\rho_1\xi - C_2\sin\rho_1\xi) \\ b_2 &= \rho_1(B_1\sin\rho_2\xi + C_1\cos\rho_2\xi) - \frac{g}{\rho_1}(B_2\sin\rho_1\xi + C_2\cos\rho_1\xi) \end{aligned} \tag{39}$$

where B_i, C_i are integration constants,

$$\rho_{1,2} = \tfrac{1}{2}(1 \pm \surd(1+4g^2)) \tag{40}$$

and we have

$$K = \surd(1+4g^2)\left(\rho_1(B_1^2+C_1^2) + \frac{1}{\rho_1}(B_2^2+C_2^2)\right). \tag{41}$$

If the wave (38) is taken literally as an infinitely extended one, the particle is never separated from the laser beam and we again have to omit the factor $\exp(-iu_0 p_u^f)$ in the solution (21). The integration constants have then to be fixed by assuming prescribed values for c_i, b_i on some null plane inside the laser beam. The resulting integration constants depend on g in such a way that the solution tends smoothly to the Volkov solution for $g \to 0$ (we have anticipated this behaviour, including a factor g in the appropriate places). It has, however, to be observed that the assumption of an infinitely extended laser wave is quite unrealistic. In actual situations the particles should always be separated from the laser beam in some region, a situation which we can imitate only by a finite pulse.

5. Conclusion

The structure of the solutions, which we have found, resembles the corresponding one of the Volkov solution for a particle without anomalous moment, the only difference being a spin factor. In particular it is clear from equation (32) that the density j_v is the same as for a free particle. In order to observe effects of the laser field one has to investigate processes like eg the Compton effect. For the corresponding theory the wavefunction as derived here is needed. It should, however, be observed that the characteristic laser effects (like the generation of harmonics or the intensity-dependent frequency shift) will be much smaller than for electrons, since the magnitude of these effects is determined by (ea/mc^2). Thus it is very doubtful whether the influence of the magnetic coupling term on these effects can be made visible experimentally.

Appendix

We have assumed that the particle in its interaction with the laser field can be characterized by its static charge and anomalous magnetic moment alone, so that a corresponding Dirac equation can be used. This is, in fact, an approximation valid for comparatively low frequencies of the external field. The approximation is most easily exhibited in terms of the Green function of the particle in the external field

$$G(x, x') = \langle x|G|x'\rangle.$$

Its inverse G^{-1}, considered as an operator in coordinate space, is the differential operator which, acting on the wavefunction, yields the corresponding wave equation. An expansion of G^{-1} for low frequencies was developed by Klein (1955) on the assumption that strong interactions may be described by a renormalizable field theory. It is based upon the invariance against Lorentz transformations, gauge transformations and charge conjugation, which fixes the form of G^{-1} in the neighbourhood of the mass shell (in which we are interested, since we want an equation for the wavefunction) to a large extent. The remaining constants are identified by phenomenological considerations. The result reads, with $\pi^\mu = P^\mu - eA^\mu$ and the anomalous moment μ_A

$$G^{-1} = \gamma_\mu \pi^\mu - \kappa - \frac{\mu_A}{2}\sigma_{\mu\nu}F^{\mu\nu} + \dots$$

which is the expression we have used for our wave equation. The next term in the expansion would contain higher powers of the field strength in the form of the combinations (F^* is the dual field tensor)

$$\rho F_{\mu\nu}F^{\mu\nu}, \qquad \sigma\gamma_5 F_{\mu\nu}F^{*\mu\nu}, \qquad \frac{\lambda}{\kappa}\gamma_\mu \pi_\nu F^{\mu\lambda}F_\lambda{}^\nu,$$

where ρ, σ, λ are constants. In our case only the last term is different from zero. In the non-relativistic limit its contribution to G^{-1} becomes proportional to $\lambda \boldsymbol{E}^2$, where λ can be understood as an electric polarizability of the nucleon, which is phenomenologically determined as

$$\lambda \sim \frac{1}{\kappa}\left(\frac{e}{m_\pi c^2}\right)^2$$

where m_π is the pion mass. Thus the term has in our case a factor

$$\frac{ea}{m_\pi c^2}\frac{\omega}{\kappa_\pi}$$

in comparison with the contribution from the magnetic moment and can therefore safely be neglected.

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